

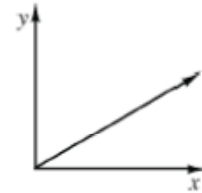
CHAPTER 1



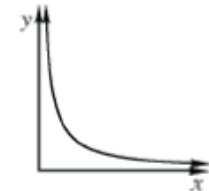
METHODS AND MEANINGS

Families of Functions

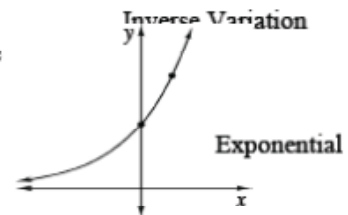
There are several “families” of special functions that you will study in this course. One of these is called **direct variation** (also called **direct proportion**) which is a **linear** function. The data you gathered in the “Sign on the Dotted Line” lab (in problem 1-9) is an example of a linear function.



Another function is **inverse variation** (also called **inverse proportion**). The data collected in the “Hot Tub Design” lab (in problem 1-9) is an example of inverse variation.



You also observed an **exponential** function. The growth of infected people in the “Local Crisis” (in problem 1-9) was **exponential**.



Note that we will define and develop these and other functions later in the course, and formally introduce functions in Section 2 of this chapter.

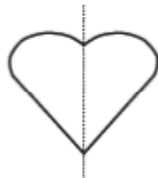


MATH NOTES

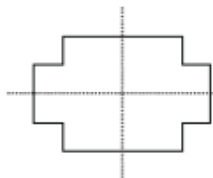
METHODS AND MEANINGS

Lines of Symmetry

When a graph or picture can be folded so that both sides of the fold will perfectly match, it is said to have **reflective symmetry**. The line where the fold would be is called the **line of symmetry**. Some shapes have more than one line of symmetry. See the examples below.



This shape has one line of symmetry.



This shape has two lines of symmetry.



This shape has eight lines of symmetry.



This graph has two lines of symmetry.



This shape has no lines of symmetry.



MATH NOTES

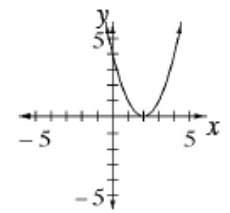
METHODS AND MEANINGS

Functions

A relationship between inputs and outputs is a **function** if there is no more than one output for each input. We often write a function as $y =$ some expression involving x , where x is the input and y is the output. The following is an example of a function.

$$y = (x - 2)^2$$

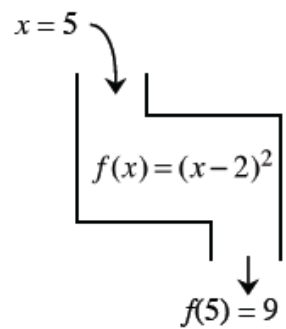
x	-2	-1	0	1	2	3	4	5
y	16	9	4	1	0	1	4	9



In the example above the value of y depends on x , so y is also called the **dependent variable** and x is called the **independent variable**.

Another way to write a function is with the notation " $f(x) =$ " instead of " $y =$ ". The function named " f " has output $f(x)$. The input is x .

In the example at right, $f(5) = 9$. The input is 5 and the output is 9. You read this as, " f of 5 equals 9."

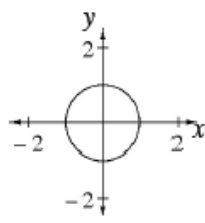


The set of all inputs for which there is an output is called the **domain**. The set of all possible outputs is called the **range**. In the example above, notice that you can input any x -value into the equation and get an output. The domain of this function is "all real numbers" because any number can be an input. But the outputs are all greater than or equal to zero. The range is $y \geq 0$.

$x^2 + y^2 = 1$ is not a function because there are two y -values (outputs) for some x -values, as shown below.

$$x^2 + y^2 = 1$$

x	-1	0	0	1
y	0	-1	1	0





METHODS AND MEANINGS

The **slope** of a line is the ratio of the vertical distance to the horizontal distance of a slope triangle formed by two points on a line. The vertical part of the triangle is called Δy (read “change in y ”), while the horizontal part of the triangle is called Δx (read “change in x ”). It indicates both how steep the line is and its direction, upward or downward, left to right.

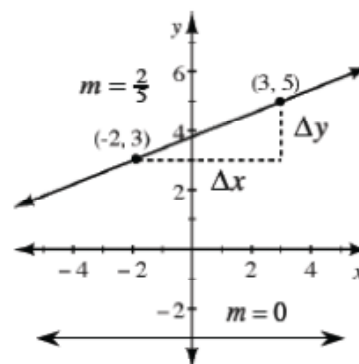
Note that “ Δ ” is the Greek letter “delta” that is often used to represent a difference or a change.

Note that lines pointing upward from left to right have positive slope, while lines pointing downward from left to right have negative slope. A horizontal line has zero slope, while a vertical line has undefined slope.

To calculate the slope of a line, pick two points on the line, draw a slope triangle (as shown in the example above), determine Δy and Δx , and then write the slope ratio. You can verify that your slope correctly resulted in a negative or positive value based on its direction. In the example above, $\Delta y = 2$ and $\Delta x = 5$, so the slope is $\frac{2}{5}$.

The Slope of a Line

$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\Delta y}{\Delta x}$$





MATH NOTES

METHODS AND MEANINGS

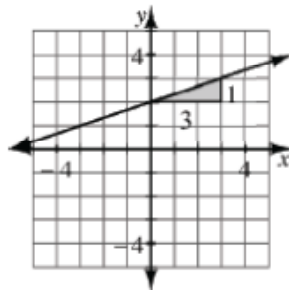
Writing the Equation of a Line from a Graph

One of the ways to write the equation of a line directly from a graph is to find the slope of the line (m) and the y -intercept (b). These values can then be substituted into the general slope-intercept form of a line: $y = mx + b$.

For example, the slope of the line at right is $m = \frac{1}{3}$, while the y -intercept is $(0, 2)$. By substituting $m = \frac{1}{3}$ and $b = 2$ into $y = mx + b$, the equation of the line is:

$$y = mx + b \rightarrow y = \frac{1}{3}x + 2$$

↑ slope ↑ y -intercept

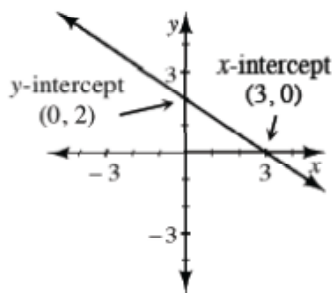


MATH NOTES

METHODS AND MEANINGS

x - and y - Intercepts

Recall that the x -intercept of a line is the point where the graph crosses the x -axis (where $y = 0$). To find the x -intercept, substitute 0 for y and solve for x . The coordinates of the x -intercept are $(x, 0)$.



Similarly, the y -intercept of a line is the point where the graph crosses the y -axis, which happens when $x = 0$. To find the y -intercept, substitute 0 for x and solve for y . The coordinates of the y -intercept are $(0, y)$.

Example: The graph of $2x + 3y = 6$ is a line, as shown above right.

To calculate the x -intercept,
let $y = 0$: $2x + 3(0) = 6$

$$2x = 6$$

$$x = 3$$

x -intercept: $(3, 0)$

To calculate the y -intercept,
let $x = 0$: $2(0) + 3y = 6$

$$3y = 6$$

$$y = 2$$

y -intercept: $(0, 2)$



METHODS AND MEANINGS

Laws of Exponents

In the expression x^3 , x is the **base** and 3 is the **exponent**.

$$x^3 = x \cdot x \cdot x$$

The patterns that you have been using during this section of the book are called the **laws of exponents**. Here are the basic rules with examples:

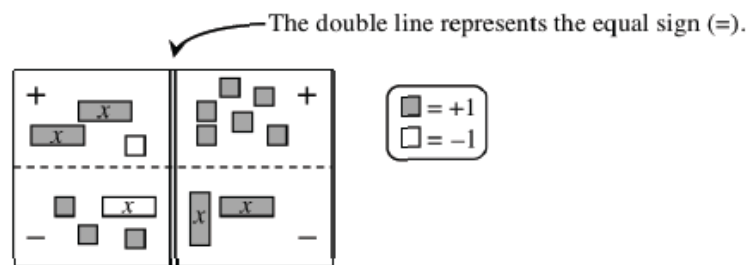
Law	Examples
$x^m x^n = x^{m+n}$ for all x	$x^3 x^4 = x^{3+4} = x^7$ $2^5 \cdot 2^{-1} = 2^4$
$\frac{x^m}{x^n} = x^{m-n}$ for $x \neq 0$	$x^{10} \div x^4 = x^{10-4} = x^6$ $\frac{5^4}{5^7} = 5^{-3}$
$(x^m)^n = x^{mn}$ for all x	$(x^4)^3 = x^{4 \cdot 3} = x^{12}$ $(10^5)^6 = 10^{30}$
$x^0 = 1$ for $x \neq 0$	$\frac{y^2}{y^2} = y^0 = 1$ $9^0 = 1$
$x^{-1} = \frac{1}{x}$ for $x \neq 0$	$\frac{1}{x^2} = \left(\frac{1}{x}\right)^2 = (x^{-1})^2 = x^{-2}$ $3^{-1} = \frac{1}{3}$



METHODS AND MEANINGS

Using Algebra Tiles to Solve Equations

Algebra tiles are a physical and visual representation of an equation. For example, the equation $2x + (-1) - (-x) - 3 = 6 - 2x$ can be represented by the Equation Mat below.



For each side of the equation, there is an addition and a subtraction region.

An Equation Mat can be used to represent the process of solving an equation. The “legal” moves on an Equation Mat correspond with the mathematical properties used to algebraically solve an equation.

“Legal” Tile Move

Group tiles that are alike together.

Flip all tiles from subtraction region to addition region.

Flip everything on both sides.

Remove zero pairs (pairs of tiles that are opposites) within a region of the mat.

Place or remove the same tiles on or from both sides.

Arrange tiles into equal-sized groups.

Corresponding Algebra

Combine like terms.

Change subtraction to “adding the opposite.”

Multiply (or divide) both sides by -1 .

A number plus its opposite equals zero.

Add or subtract the same value from both sides.

Divide both sides by the same value.



METHODS AND MEANINGS

Multiplying Algebraic Expressions with Tiles

The area of a rectangle can be written two different ways. It can be written as a *product* of its width and length or as a *sum* of its parts. For example, the area of the shaded rectangle at right can be written two ways:

x				
x				
x^2	x	x	x	x

$$\underbrace{(x+4)}_{\text{length}} \underbrace{(x+2)}_{\text{width}} = \underbrace{x^2+6x+8}_{\text{area}}$$

area as a product = area as a sum



METHODS AND MEANINGS

Vocabulary for Expressions

A mathematical **expression** is a combination of numbers, variables, and operation symbols. Addition and subtraction separate expressions into parts called **terms**. For example, $4x^2 - 3x + 6$ is an expression. It has three terms: $4x^2$, $3x$, and 6 . The **coefficients** are 4 and 3 . 6 is called a **constant term**.

A one-variable **polynomial** is an expression which only has terms of the form:

$$(\text{any real number})x^{(\text{whole number})}$$

For example, $4x^2 - 3x^1 + 6x^0$ is a polynomial, so the simplified form, $4x^2 - 3x + 6$ is a polynomial.

The function $f(x) = 7x^5 + 2.5x^3 - \frac{1}{2}x + 7$ is a polynomial function.

The following are not polynomials: $2^x - 3$, $\frac{1}{x^2-2}$, and $\sqrt{x-2}$.

A **binomial** is a polynomial with only two terms, for example, $x^3 - 0.5x$ and $2x + 5$.



METHODS AND MEANINGS

Properties of Real Numbers

The legal tiles moves have formal mathematical names, called the **properties of real numbers**.

The **Commutative Property** states that when *adding* or *multiplying* two or more number or terms, order is not important. That is:

$$a + b = b + a \quad \text{For example, } 2 + 7 = 7 + 2$$

$$a \cdot b = b \cdot a \quad \text{For example, } 3 \cdot 5 = 5 \cdot 3$$

However, *subtraction* and *division* are not commutative, as shown below.

$$7 - 2 \neq 2 - 7 \quad \text{since } 5 \neq -5$$

$$50 \div 10 \neq 10 \div 50 \quad \text{since } 5 \neq 0.2$$

The **Associative Property** states that when *adding* or *multiplying* three or more number or terms together, grouping is not important. That is:

$$(a + b) + c = a + (b + c) \quad \text{For example, } (5 + 2) + 6 = 5 + (2 + 6)$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \text{For example, } (5 \cdot 2) \cdot 6 = 5 \cdot (2 \cdot 6)$$

However, *subtraction* and *division* are not associative, as shown below.

$$(5 - 2) - 3 \neq 5 - (2 - 3) \quad \text{since } 0 \neq 6$$

$$(20 \div 4) \div 2 \neq 20 \div (4 \div 2) \quad \text{since } 2.5 \neq 10$$

The **Identity Property of Addition** states that adding zero to any expression gives the same expression. That is:

$$a + 0 = a \quad \text{For example, } 6 + 0 = 6$$

The **Identity Property of Multiplication** states that multiplying any expression by one gives the same expression. That is:

$$1 \cdot a = a \quad \text{For example, } 1 \cdot 6 = 6$$

The **Additive Inverse Property** states that for every number a there is a number $-a$ such that $a + (-a) = 0$. A common name used for the additive inverse is the **opposite**. That is, $-a$ is the opposite of a . For example, $3 + (-3) = 0$ and $-5 + 5 = 0$.

The **Multiplicative Inverse Property** states that for every nonzero number a there is a number $\frac{1}{a}$ such that $a \cdot \frac{1}{a} = 1$. A common name used for the multiplicative inverse is the **reciprocal**. That is, $\frac{1}{a}$ is the reciprocal of a . For example, $6 \cdot \frac{1}{6} = 1$.



MATH NOTES

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The Distributive Property

The **Distributive Property** states that for any three terms a , b , and c :

$$a(b + c) = ab + ac$$

That is, when a multiplies a group of terms, such as $(b + c)$, then it multiplies *each* term of the group. For example, when multiplying $2x(3x + 4y)$, the $2x$ multiplies both the $3x$ and the $4y$. This can be shown with **algebra tiles** or in a **generic rectangle** (see below).

	x	x	x	y	y	y	y
x	x^2	x^2	x^2	xy	xy	xy	xy
x	x^2	x^2	x^2	xy	xy	xy	xy

$$2x \begin{array}{|c|c|} \hline 2x \cdot 3x & 2x \cdot 4y \\ \hline 3x & 4y \\ \hline \end{array}$$

$$2x(3x + 4y) = 2x(3x) + 2x(4y), \text{ simplifying results in } 6x^2 + 8xy$$


 The $2x$ multiplies each term.



METHODS AND MEANINGS

Linear Equations from Slope and/or Points

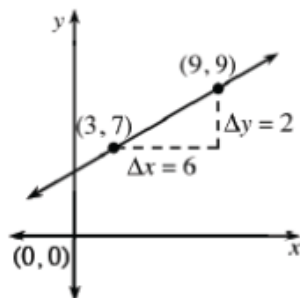
If you know the slope, m , and y-intercept, $(0, b)$, of a line, you can write the equation of the line as $y = mx + b$.

You can also find the equation of a line when you know the slope and one point on the line. To do so, rewrite $y = mx + b$ with the known slope and substitute the coordinates of the known point for x and y . Then solve for b and write the new equation.

For example, find the equation of the line with a slope of -4 that passes through the point $(5, 30)$. Rewrite $y = mx + b$ as $y = -4x + b$. Substituting $(5, 30)$ into the equation results in $30 = -4(5) + b$. Solve the equation to find $b = 50$. Since you now know the slope and y-intercept of the line, you can write the equation of the line as $y = -4x + 50$.

Similarly you can write the equation of the line when you know two points. First use the two points to find the slope. Then substitute the known slope and either of the known points into $y = mx + b$. Solve for b and write the new equation.

For example, find the equation of the line through $(3, 7)$ and $(9, 9)$. The slope is $\frac{\Delta y}{\Delta x} = \frac{2}{6} = \frac{1}{3}$. Substituting $m = \frac{1}{3}$ and $(x, y) = (3, 7)$ into $y = mx + b$ results in $7 = \frac{1}{3}(3) + b$. Then solve the equation to find $b = 6$. Since you now know the slope and y-intercept, you can write the equation of the line as $y = \frac{1}{3}x + 6$.



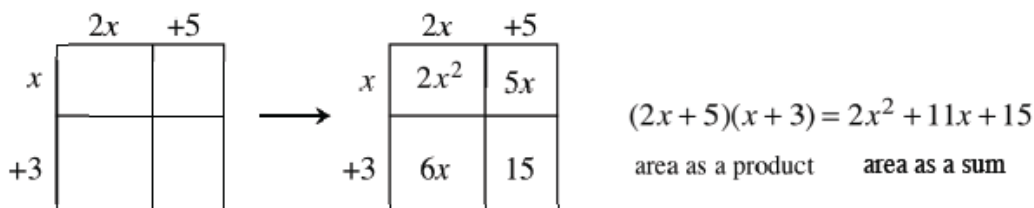


MATH NOTES

METHODS AND MEANINGS

Using Generic Rectangles to Multiply

A generic rectangle can be used to find products because it helps to organize the different areas that make up the total rectangle. For example, to multiply $(2x + 5)(x + 3)$, a generic rectangle can be set up and completed as shown below. Notice that each product in the generic rectangle represents the area of that part of the rectangle.



Note that while a generic rectangle helps organize the problem, its size and scale are not important. Some students find it helpful to write the dimensions on the rectangle twice, that is, on both pairs of opposite sides.



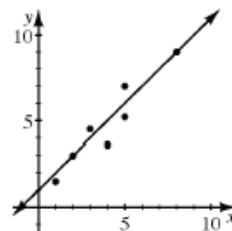
MATH NOTES

METHODS AND MEANINGS

A **line of best fit** is a straight line that represents, in general, data on a scatterplot, as shown in the diagram. This line does not need to touch any of the actual data points, nor does it need to go through the origin. The line of best fit is a model of numerical two-variable data that helps describe the data in order to make predictions for other data.

To write the equation of a line of best fit, find the coordinates of any two convenient points on the line (they could be lattice points where the gridlines intersect, or they could be data points, or the origin, or a combination). Then write the equation of the line that goes through these two points.

Line of Best Fit





MATH NOTES

METHODS AND MEANINGS

The Equal Values Method

The **Equal Values Method** is a method to find the solution to a system of equations. For example, solve the system of equations below:

$$2x + y = 5$$

$$y = x - 1$$

Put both equations into $y = mx + b$ form. The two equations are now $y = -2x + 5$ and $y = x - 1$.

$$2x + y = 5 \quad | \quad y = x - 1$$

$$y = -2x + 5$$

Take the two expressions that equal y and set them equal to each other. Then solve this new equation to find x . See the example at right.

$$-2x + 5 = x - 1$$

$$-3x = -6$$

$$x = 2$$

Once you know x , substitute your solution for x into *either* original equation to find y . In this example, the second equation is used.

$$y = x - 1$$

$$y = (2) - 1$$

$$y = 1$$

Check your solution by evaluating for x and y in *both* of the *original* equations.

$$2x + y = 5 \quad | \quad y = x - 1$$

$$2(2) + 1 = 5 \quad | \quad 1 = 2 - 1$$

$$5 = 5 \quad | \quad 1 = 1$$

$$\checkmark \quad | \quad \checkmark$$

The solution is $x = 2$ and $y = 1$.



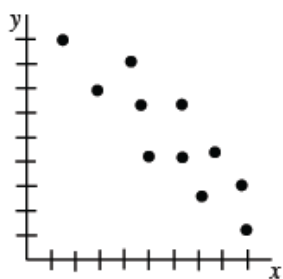
METHODS AND MEANINGS

Describing Association

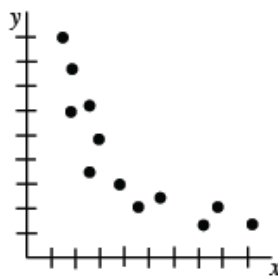
An association (relationship) between two numerical variables can be described by its form, direction, strength, and outliers.

The shape of the pattern is called the **form** of the association: **linear** or **non-linear**. The form can be made of **clusters** of data.

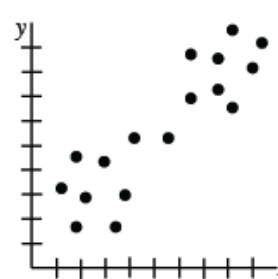
If one variable increases as the other variable increases, the **direction** is said to be a **positive association**. If one variable increases as the other variable decreases, there is said to be a **negative association**. If there is no apparent pattern in the scatterplot, then the variables have **no association**.



Negative **linear** association

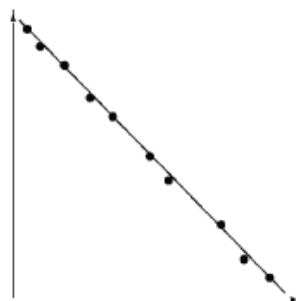


Negative **non-linear** association

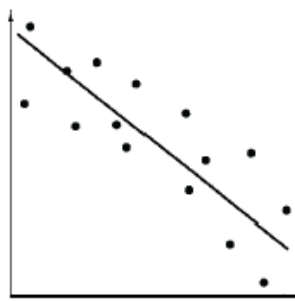


Positive **linear** association with **clusters**

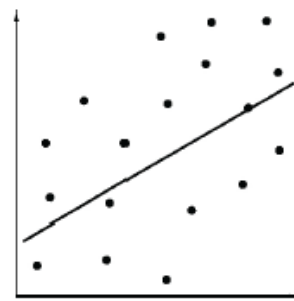
Strength is a description of how much scatter there is in the data away from the line of best fit. See some examples below.



strong negative association

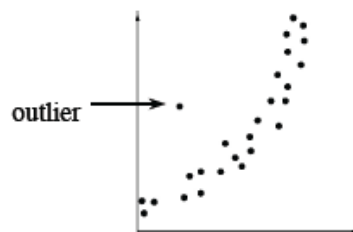


moderate negative association



weak positive association

An **outlier** is a piece of data that does not seem to fit into the overall pattern. There is one obvious outlier in the association graphed at right.





METHODS AND MEANINGS

The Substitution Method

The **Substitution Method** is a way to change two equations with two variables into one equation with one variable. It is convenient to use when only one equation is solved for a variable.

For example, to solve the system: $x = -3y + 1$
 $4x - 3y = -11$

Use substitution to rewrite the two equations as one. In other words, replace x in the second equation with $(-3y + 1)$ from the first equation to get $4(-3y + 1) - 3y = -11$. This equation can then be solved to find y . In this case, $y = 1$.

To find the point of intersection, substitute the value you found into either original equation to find the other value.

In the example, substitute $y = 1$ into $x = -3y + 1$ and write the answer for x and y as an ordered pair.

To check the solution, substitute $x = -2$ and $y = 1$ into *both* of the *original* equations.

$$\begin{aligned}x &= -3y + 1 \\4(-3y + 1) - 3y &= -11 \\4(-3y + 1) - 3y &= -11 \\-12y + 4 - 3y &= -11 \\-15y + 4 &= -11 \\-15y &= -15 \\y &= 1 \\x &= -3(1) + 1 \\x &= -2 \\(-2, 1)\end{aligned}$$

$$\begin{aligned}-2 &= -3(1) + 1 & 4(-2) - 3(1) &= -11 \\-2 &= -2 & -11 &= -11\end{aligned}$$



METHODS AND MEANINGS

Systems of Linear Equations

A **system of linear equations** is a set of two or more linear equations that are given together, such as the example at right:

$$y = 2x$$

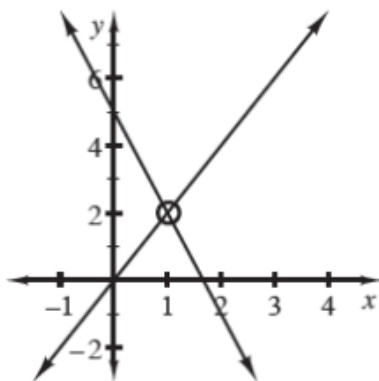
$$y = -3x + 5$$

If the equations come from a real-world situation, then each variable will represent some type of quantity in both equations. For example, in the system of equations above, y could stand for a number of dollars in *both* equations and x could stand for the number of weeks.

To represent a system of equations graphically, you can simply graph each equation on the same set of axes. The graph may or may not have a **point of intersection**, as shown circled at right.

Also notice that the point of intersection lies on *both* graphs in the system of equations. This means that the point of intersection is a **solution** to *both* equations in the system. For example, the point of intersection of the two lines graphed above is $(1, 2)$. This point of intersection makes both equations true, as shown at right.

The point of intersection makes both equations true; therefore the point of intersection is a solution to both equations. For this reason, the point of intersection is sometimes called a **solution to the system of equations**.



$$y = 2x$$

$$(2) = 2(1)$$

$$2 = 2$$

$$y = -3x + 5$$

$$(2) = -3(1) + 5$$

$$2 = -3 + 5$$

$$2 = 2$$



MATH NOTES

METHODS AND MEANINGS

Forms of a Linear Function

There are three main forms of a linear function: slope-intercept form, standard form, and point-slope form. Study the examples below.

Slope-Intercept form: $y = mx + b$. The slope is m , and the y -intercept is $(0, b)$.

Standard form: $ax + by = c$

Point-Slope form: $y - k = m(x - h)$. The slope is m , and (h, k) is a point on the line. For example, if the slope is -7 and the point $(10, 20)$ is on the line, the equation of the line can be written $y - (-10) = -7(x - 20)$ or $y + 10 = -7(x - 20)$.



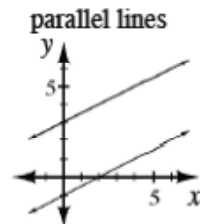
MATH NOTES

METHODS AND MEANINGS

Intersection, Parallel, and Coincide

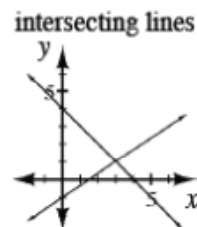
When two lines lie on the same flat surface (called a plane), they may **intersect** (cross each other) once, an infinite number of times, or never.

For example, if the two lines are **parallel**, then they never intersect. The system of these two lines does not have a solution. Examine the graph of two parallel lines at right. Notice that the distance between the two lines is constant and that they have the same slope but different y -intercepts.



However, what if the two lines lie exactly on top of each other? When this happens, we say that the two lines **coincide**. When you look at two lines that coincide, they appear to be one line. Since these two lines intersect each other at all points along the line, coinciding lines have an infinite number of intersections. The system has an infinite number of solutions. Both lines have the same slope and y -intercept.

While some systems contain lines that are parallel and others coincide, the most common case for a system of equations is when the two lines intersect once, as shown at right. The system has one solution, namely, the point where the lines intersect, (x, y) .





METHODS AND MEANINGS

The Elimination Method

When solving a system of equations, it may be easier to eliminate one of the variables by adding multiples of the two equations. This process is called **elimination**.

The first step is to rewrite the equations so that the x and y variables are lined up vertically. Next, decide what number to multiply each equation by, if necessary, in order to make the coefficients of either the x -terms or the y -terms add up to zero. Be sure that you can justify each step in the solution.

For example, consider the system at right.

$$10y - 3x = 14$$

$$4y + 2x = -4$$

You can eliminate the x -terms by multiplying the top equation by 2 and the bottom equation by 3 and then adding the equations, as shown at right.

$$(10y - 3x = 14) \cdot 2 \rightarrow 20y - 6x = 28$$

$$(4y + 2x = -4) \cdot 3 \rightarrow \underline{12y + 6x = -12}$$

$$\text{Add the resulting equations: } 32y = 16$$

$$\text{Divide: } y = 0.5$$

Finally, substitute 0.5 for y in either original equation:

$$10(0.5) - 3x = 14$$

Thus, the solution to the original system is $(-3, 0.5)$.

$$5 - 3x = 14$$

$$-3x = 9$$

Check your solution by evaluating for x and y in *both* of the *original* equations.

$$x = -3$$

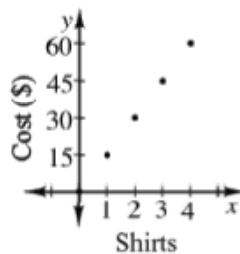


METHODS AND MEANINGS

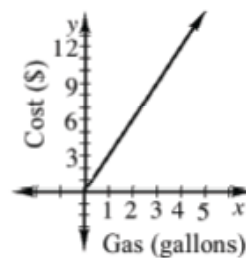
Continuous and Discrete Graphs

When the points on a graph are connected, and it *makes sense* to connect them, the graph is said to be **continuous**. If the graph is not continuous, and is just a sequence of separate points, the graph is called **discrete**. For example, the graph below left represents the cost of buying x shirts, and it is discrete because you can only buy whole numbers of shirts. The graph farthest right represents the cost of buying x gallons of gasoline, and it is continuous because you can buy any (non-negative) amount of gasoline.

Discrete Graph



Continuous Graph





METHODS AND MEANINGS

Types of Sequences

An **arithmetic sequence** is a sequence with an addition (or subtraction) **sequence generator**. The number added to each term to get the next term is called the **common difference**.

A **geometric sequence** is a sequence with a multiplication (or division) generator. The number multiplied by each term to get the next term is called the **common ratio** or the **multiplier**.

A multiplier can also be used to increase or decrease by a given percentage. For example, the multiplier for an increase of 7% is 1.07. The multiplier for a decrease of 7% is 0.93.

A **recursive sequence** is a sequence in which each term depends on the term(s) before it. The equation of a recursive sequence requires at least one term to be specified. A recursive sequence can be arithmetic, geometric, or neither.

For example, the sequence $-1, 2, 5, 26, 677, \dots$ can be defined by the **recursive equation**:

$$t(1) = -1, \quad t(n+1) = (t(n))^2 + 1$$

An alternative notation for the equation of the sequence above is:

$$a_1 = -1, \quad a_{n+1} = (a_n)^2 + 1$$